Int. J. Solids Structures Vol. 14, pp. 935–939
© Pergamon Press Ltd., 1978. Printed in Great Britain

0020-7683/78/1101-0935/$02.00/0

SOME REMARKS ABOUT THE FORMULATION OF THREE-DIMENSIONAL THERMOELASTOPLASTIC PROBLEMS BY INTEGRAL EQUATIONS

H. D. Bui

Centre des Renardières, Electricité de France, Laboratoire de Mécanique des Solides, Ecole Polytechnique, Palaiseau 91128, France

(*Received* 13 *January* 1978; *in revised form* 23 *May* 1978)

Abstract—The extended form of the identity of Somigliana, giving the displacement rate in terms of the traction and the displacement rate on the boundary, involves plastic and thermal strain rate. Because of strong singularity of some kernel, the gradient of the displacement rate may be strictly evaluated by an integro-differential expression. The paper shows how to derive the integral form of the gradient rate which corrects some erroneous expressions reported in the literature.

INTRODUCTION

The boundary integral equation method to solve the three-dimensional problems of isotropic elastic solids has been extended to the problems of plastic flow[1, 2] and thermal loading[2, 3]. For given plastic deviatoric strain rate $\dot{\epsilon}^p_{ij}$ and thermal strain rate $\dot{\epsilon}^T_{ij} = \alpha\dot{T}\delta_{ij}$, the displacement rate $\dot{u}_i(p)$ at any point $p = (p_1, p_2, p_3)$ satisfies the Somigliana identity[1, 2]

$$\gamma(p)\dot{u}_k(p) = -\int_{\partial\Omega} T_{ki}(p, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q + \int_{\partial\Omega} U_{ki}(p, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q + \int_\Omega U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q + \int_\Omega \Sigma_{kij}(p, q)(\dot{\epsilon}^p_{ij}(q) + \dot{\epsilon}^T_{ij}(q))\,\mathrm{d}v_q \quad (1)$$

where $\dot{t}_i$ is the traction rate $\dot{t}_i = \dot{\sigma}_{ij}n_j$ on the boundary $\partial\Omega$, $\dot{f}_i$ is the prescribed body force rate per unit volume, $U_{ki}(p, q)$ is the Kelvin–Somigliana tensor[1, 2, 4]:

$$U_{ki}(p, q) = \frac{1}{16\pi\mu(1-\nu)r(p, q)}\left\{(3-4\nu)\delta_{ik} + \frac{(p_k - q_k)(p_i - q_i)}{r^2}\right\} \quad (2)$$

$r(p, q)$ being the distance between the points p and q, μ is the shear modulus and ν is the Poisson ratio, $\Sigma_{kij}(p, q)$ is the stress tensor corresponding to $U_{ki}(p, q)$:

$$\Sigma_{kij}(p, q) = -\frac{1}{8\pi(1-\nu)r^2}\left\{(1-2\nu)\left(\delta_{ki}\frac{\partial r}{\partial q_j} + \delta_{kj}\frac{\partial r}{\partial q_i} - \delta_{ij}\frac{\partial r}{\partial q_k}\right) + 3\frac{\partial r}{\partial q_k}\frac{\partial r}{\partial q_i}\frac{\partial r}{\partial q_j}\right\}. \quad (3)$$

The traction vector corresponding to Σ_{kij} is $T_{ki} = \Sigma_{kij}n_j$.

In eqn (1) we put $\gamma = 1$ for interior points and $\gamma = 1/2$ for points belonging to the smooth boundary.† The value $\gamma = 1/2$ corresponds to the boundary constraint equation. The equation for $\gamma = 1$ is referred to the field equation. The constraint equation together with the field equation and the constitutive equation[1]:

$$\dot{\epsilon}^p_{ij} = \begin{cases} A_{ijhk}\dot{u}_{h,k} & \text{for plastic loading} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

constitute the basic nonlinear system of equations used to solve the boundary value problem. Generally, an iterative scheme is used to solve these equations. First, one may determine the solution of the boundary constraint equation for some plastic strain rate at some stage of the iteration, then the gradient $\dot{u}_{h,k}$ is calculated from the field equation and from (4) one determines

†The case of singular boundary is discussed in Ref.[4].

the plastic strain rate for the next stage. This procedure is well known in the literature. However a difficulty may arise when we want to calculate the gradient field. Of course, the correct definition is given by:

$$\dot{u}_{k,h}(p) = -\int_{\partial\Omega} \frac{\partial}{\partial p_h} T_{ki}(p, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q + \int_{\partial\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q + \int_{\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q + \frac{\partial}{\partial p_h}\int_{\Omega} \Sigma_{kij}(p, q)(\dot{\epsilon}^p_{ij}(q) + \dot{\epsilon}^T_{ij}(q))\,\mathrm{d}v_q. \tag{5}$$

Differentiation has been applied to the kernels of the surface integrals. This is permissible because the kernels $\partial T_{ki}/\partial p_h$ and $\partial U_{ki}/\partial p_h$ are regular at interior points, $r(p, q) \neq 0$, while the kernel $\partial U_{ki}/\partial p_h$ of order $0(r^{-2})$ is integrable in Ω. Unfortunately, differentiation under the fourth volume integral sign is *not* allowable, because of strong singularity of the kernel $\partial \Sigma_{kij}/\partial p_h$ the behaviour of which is of order of $0(r^{-3})$ as $r \to 0$. Differentiation of such an integral does not obey to classical rule [5]. This fact has been ignored in the literature [2, 6]. The aim of this paper is to derive a correct expression for the gradient rate. Applications will be given to the plastic inclusion problem of Eshelby [7] and to the thermal inclusion problem.

CONVECTED DIFFERENTIATION OF SINGULAR INTEGRAL

It is obvious that the kernel $\Sigma_{kij}(p, q)$ has an integrable singularity in Ω. Let us show more precisely that the volume integral

$$V(p) = \int_{B_\eta(p)} \Sigma_{kij}(p, q)\epsilon^a_{ij}(q)\,\mathrm{d}v_q \tag{6}$$

where $\epsilon^a = \dot{\epsilon}^p + \dot{\epsilon}^T$, taken over the ball $B_\eta(p)$ of small radius η, centered at the point p, is of order $0(\eta^2)$. To prove it, we assume $\epsilon^a_{ij}(q)$ and its first and second derivatives to be continuous in the neighbourhood of the point p. Let ϵ^a be expanded as follows:

$$\epsilon^a_{ij}(q) = \epsilon^a_{ij}(p) + (q_h - p_h)\epsilon^a_{ij,h}(p) + \cdots \tag{7}$$

Hence

$$V(p) = \epsilon^a_{ij}(p)\int_{B_\eta(p)} \Sigma_{kij}(p, q)\,\mathrm{d}v_q + \epsilon^a_{ij,h}(p)\int_{B_\eta(p)} (q_h - p_h)\Sigma_{kij}(p, q)\,\mathrm{d}v_q. \tag{8}$$

Using (3), the integrals of (8) are easily computed and given by:

$$\int_{B_\eta(p)} \Sigma_{kij}(p, q)\,\mathrm{d}v_q = 0$$

$$\int_{B_\eta(p)} (q_h - p_h)\Sigma_{kij}(p, q)\,\mathrm{d}v_q = -C\eta^2 \tag{9}$$

where the constant C is

$$C = \frac{1-2\nu}{12(1-\nu)}(\delta_{ki}\delta_{jh} + \delta_{kj}\delta_{ih} - \delta_{ij}\delta_{kh}) + \frac{1}{20(1-\nu)}(\delta_{kh}\delta_{ij} + \delta_{ih}\delta_{jk} + \delta_{jh}\delta_{ik}).$$

Thus, the proof is complete. It is interesting to notice that (9) is independent of p. Hence, the gradient $\partial V/\partial p_m = -C\eta^2\epsilon^a_{ij,hm}$ is also of order $0(\eta^2)$. We now write the field equation in the following form:

$$\dot{u}_k(p) = -\int_{\partial\Omega} T_{ki}(p, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q + \int_{\partial\Omega} U_{ki}(p, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q + \int_{\Omega} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q + V(p) + \int_{\Omega - B_\eta(p)} \Sigma_{kij}(p, q)(\dot{\epsilon}^p_{ij}(q) + \dot{\epsilon}^T_{ij}(q))\,\mathrm{d}v_q \tag{10}$$

containing the integrals which can be differentiated under the integral sign, together with the term $V(p)$. After differentiation of (10), to within $0(\eta^2)$, we drop the term $\partial V/\partial p_h$ to obtain

$$\dot{u}_{k,h}(p) = -\int_{\partial\Omega} \frac{\partial}{\partial p_h} T_{ki}(p, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q + \int_{\partial\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q$$
$$+ \int_{\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q + \int_{\Omega - B_\eta(p)} \frac{\partial}{\partial p_h} \Sigma_{kij}(p, q)(\dot{\epsilon}^p_{ij}(q) + \dot{\epsilon}^T_{ij}(q))\,\mathrm{d}v_q$$
$$- \int_{\partial B_\eta} \Sigma_{kij}(p, Q)(\dot{\epsilon}^p_{ij}(Q) + \dot{\epsilon}^T_{ij}(Q))n_h(Q)\,\mathrm{d}S_Q \tag{11}$$

where n_h is the unit outward normal to the sphere ∂B_η. The fourth integral of (11), when $\eta \to 0$, yields the principal value of the singular integral the existence of which has been proved in [5, 8]. The last term over the surface of the sphere ∂B_η is nothing but the convected term due to the fact that the domain $\Omega - B_\eta(p)$ changes with the position of the point p. It is precisely the term which is ignored in the literature [2, 6].

Let us illustrate the concept of convected differentiation of singular integral by studying an elementary example. Consider the line integral

$$F(x) = \int_{-1}^{+1} \frac{\mathrm{d}t}{t - x} \equiv \log\left|\frac{1-x}{1+x}\right|.$$

Of course, its derivative is $F'(x) = (x-1)^{-1} - (x+1)^{-1}$.

Differentiation under the integral sign would be incorrect

$$\int_{-1}^{+1} \frac{\mathrm{d}t}{(t-x)^2} = \infty \qquad \text{for} \qquad |x| < 1.$$

In order to obtain the correct result, we take into account the convected terms at the end points of the interval $B_\eta(x) \equiv [x - \eta, x + \eta]$

$$F'(x) = \int_{-1}^{x-\eta} \frac{\mathrm{d}t}{(t-x)^2} + \int_{x+\eta}^{1} \frac{\mathrm{d}t}{(t-x)^2} + \frac{1}{t-x}\bigg|_{t=x-\eta} - \frac{1}{t-x}\bigg|_{t=x+\eta}.$$

Hence

$$F'(x) = \frac{2}{\eta} + \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{\eta} = \frac{1}{x-1} - \frac{1}{x+1}.$$

This example clearly shows why the differentiation of singular integral does not obey the classical rule. The classical rule may be applied to the singular integral when the convected term vanishes. For instance, there is no convected term in

$$\frac{\partial}{\partial p_h}\int_{\Omega} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q = \int_{\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q.$$

We return now to the equation (11) for which we want to calculate the convected term in a compact form. As for the calculation of (6), we make use of the expansion (7). To within the term of order of $0(\eta^2)$, by taking account of $\dot{\epsilon}^p_{ii} = 0$ and $\epsilon^T_{ij} = \alpha\dot{T}\delta_{ij}$, we find that

$$- \int_{\partial B_\eta} \Sigma_{kij}(p, Q)(\dot{\epsilon}^p_{ij}(Q) + \dot{\epsilon}^T_{ij}(Q))n_h(Q)\,\mathrm{d}S_Q = \frac{(8-10\nu)}{15(1-\nu)}\dot{\epsilon}^p_{hk}(p) + \frac{(1+\nu)}{3(1-\nu)}\alpha\dot{T}\delta_{hk}. \tag{12}$$

Consequently, the desired integral expression of the gradient rate is given by ($\eta \to 0$):

$$\dot{u}_{h,k}(p) = -\int_{\partial\Omega} \frac{\partial}{\partial p_h} T_{ki}(p, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q + \int_{\partial\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q + \int_{\Omega} \frac{\partial}{\partial p_h} U_{ki}(p, q)\dot{f}_i(q)\,\mathrm{d}v_q$$
$$+ \int_{\Omega - B_\eta} \frac{\partial}{\partial p_h} \Sigma_{kij}(p, q)(\dot{\epsilon}^p_{ij}(q) + \dot{\epsilon}^T_{ij}(q))\,\mathrm{d}v_q + \frac{8-10\nu}{15(1-\nu)}\dot{\epsilon}^p_{hk}(p) + \frac{1+\nu}{3(1-\nu)}\alpha\dot{T}\delta_{hk}. \tag{13}$$

The stress rate is then expressed as follows:

$$\dot{\sigma}_{ij} = \mu(\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{2\mu\nu}{1-2\nu}\dot{u}_{k,k}\delta_{ij} - 2\mu\dot{\epsilon}^p_{ij} - 2\mu\left(\frac{1+\nu}{1-2\nu}\right)\alpha\dot{T}\delta_{ij}.$$

The convected terms of eqn (13) and those corresponding to the stress rate cannot be neglected. For instance, with $\nu = 0.3$, the convected terms of eqn (13) are respectively equal to 0.47 $\dot{\epsilon}^p_{hk}$ and $0.62\alpha\dot{T}\delta_{hk}$. Omitting the plastic additional term in the iterative scheme would result in systematic error at each time the stress rate is calculated. We will see, through some analytical solutions, that eqn (13) without the convected terms leads to wrong results.

THE ESHELBY INCLUSION PROBLEM

Consider the particular problem of an infinite isotropic elastic body containing an elasto-plastic sphere C. The body is subject to $\dot{f}_i = 0$ and to the nullified boundary conditions at infinity. That is the displacement and the stress rate are required to decay at infinity like $\dot{u}_i = r^{-1}(0, Q)$ and $\dot{t}_i = r^{-2}(0, Q)$. Therefore, the surface integrals of (13) vanish identically. Assume now that the inclusion has a *homogeneous* plastic strain rate. We have to prove, according to the Eshelby theory, that the self equilibrated stress field, resulting from this plastic strain rate, is also homogeneous in C. It is sufficient to show that the integral of (13) extended over $C - B_\eta(p)$

$$A = \dot{\epsilon}^p_{ij}\int_{C-B_\eta}\frac{\partial}{\partial p_h}\Sigma_{kij}(p,q)\,\mathrm{d}v_q = -\dot{\epsilon}^p_{ij}\int_{C-B_\eta}\frac{\partial}{\partial q_h}\Sigma_{kij}(p,q)\,\mathrm{d}v_q$$

vanishes identically for any point p within C.

First we note that, the kernels being regular in $C - B_\eta$, the integration by parts is allowable†

$$A = -\dot{\epsilon}^p_{ij}\left\{\int_{\partial C}\Sigma_{kij}(p,q)n_h(q)\,\mathrm{d}S_q - \int_{\partial B_\eta}\Sigma_{kij}(p,q)n_h(q)\,\mathrm{d}S_q\right\} \tag{14}$$

where n_h is the unit outward normal to the sphere ∂C or ∂B_η. Remark that $\dot{\epsilon}^p_{ij}$ is a deviatoric tensor

$$\dot{\epsilon}^p_{ij}\Sigma_{kij}(p,q) = 2\mu\dot{\epsilon}^p_{ij}\frac{\partial}{\partial q_j}U_{ki}(p,q). \tag{15}$$

Next, for any sphere S containing the point p, we have the identity

$$\int_S\frac{\partial}{\partial q_j}U_{ki}(p,q)n_h(q)\,\mathrm{d}S_q = -\frac{1}{3\mu}\delta_{ik}\delta_{jh} + \frac{1}{30\mu(1-\nu)}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk} + \delta_{ij}\delta_{kh}). \tag{16}$$

Equations (14)–(16) give $A = 0$. Thus the strain rate and the stress rate in the inclusion C are homogeneous and given exactly by our convected term:

$$\dot{\epsilon}_{hk}(C) = \beta\dot{\epsilon}^p_{hk}(C) \tag{17}$$

$$\dot{\sigma}_{hk}(C) = 2\mu(\beta - 1)\dot{\epsilon}^p_{hk}(C) \tag{18}$$

where $\beta = (8 - 10\nu)/15(1-\nu)$. Formulae (17) and (18) are in accord with Eshelby's solution. Note that the coefficient β can also be found in the theories of polycristal aggregates (see Kröner[9], Budiansky and Wu[10], Bui[11]).

THE THERMAL INCLUSION PROBLEM

The problem of a spherical inclusion C embedded in an infinite elastic medium and subject to uniform thermal strain ϵ^T can be solved in elementary manner. Let us consider the infinite

†The rule of integration by parts is not generally valid for singular integrals (see Mikhlin[5]).

medium with spherical cavity of radius R. Applying the normal stress σ_{rr}^M at the cavity results in the radial displacement u^M at $r = R$:

$$u^M = -\frac{R(1+\nu)}{2E}\sigma_{rr}^M \tag{19}$$

where E denotes the Young modulus. Next, consider the uniform expansion of the sphere C of radius R, under the same normal stress σ_{rr}^M applied to its boundary, while the thermal loading is applied to it. The displacement at $r = R$ can be easily shown to be

$$u^M = \frac{R(1-2\nu)}{E}\sigma_{rr}^M + R\epsilon^T. \tag{20}$$

By equaling (19) and (20), we obtain the applied stress and the strain in the inclusion

$$\sigma_{rr}^M = -\frac{2E\epsilon^T}{3(1-\nu)}$$

$$\epsilon_{ij} = \frac{1+\nu}{3(1-\nu)}\epsilon^T\delta_{ij}. \tag{21}$$

The result (21) agrees with eqn (13) because the integrals of (13) vanish here. Once again, the example confirms that the eqn (13) without the convected terms leads to incorrect result.

CONCLUDING REMARKS

It can be concluded that correct differentiation of a singular integral yields additional terms, just like the integration by parts of a singular integral gives an extraneous term, as shown in a previous paper[8]. Due to the singular nature of the kernel and of its derivatives, the derivation of the strain rate from the field equation must be done carefully. This difficulty has already been recognized in the unpublished works[12, 13] which are brought to the author's knowledge by one reviewer.

REFERENCES

1. J. L. Swedlow and T. A. Cruse, Formulation of boundary integral equation for three-dimensional elasto-plastic body. *Int. J. Solids Structures* 7, 1673–1683 (1971).
2. A. Mendelson, Boundary integral methods in elasticity and plasticity. NASA TND-7418 (1973).
3. F. J. Rizzo and D. J. Shippy, An advanced boundary integral equation method for three-dimensional thermoelasticity. *Int. J. Num. Meth. Engng* 11, 1753–1768 (1977).
4. H. D. Bui, *Mécanique de la Rupture Fragile.* Masson, Paris (1978).
5. S. G. Mikhlin, Singular integral equations. *A.M.S.T.* 10, 84–197 (1962).
6. S. Mukherjee, Corrected boundary integral equation in planar thermoelastoplasticity. *Int. J. Solids Structures* 13, 331–335 (1977).
7. J. D. Eshelby, *Progress in Solid Mechanics* (Edited by I. N. Sneddon and R. Hill), Vol. 2. North-Holland, Amsterdam (1960).
8. H. D. Bui and K. Dangvan, Sur le problème aux limites en vitesse des contraintes du solide élasto-plastique. *Int. J. Solids Structures* 6, 183–193 (1970).
9. E. Kröner, Zur plastischen Verformung der viel kristalls. *Acta Metallurgica*, 9, 155 (1961).
10. B. Budiansky and T. T. Wu, *Proc. 4th U.S. Nat. Cong. Appl. Mech.* 1175 (1962).
11. H. D. Bui, *Extrait Mém. Artillerie Fr. Sci. Tech. Armem.* 1, 141–165 (1970).
12. P. C. Riccardella, An implementation of the boundary-integral technique for planar problems in elasticity and elasto-plasticity. Rep. No. SM-73-10, Dept. Mech. Engng, Carnegie Mellon Univ., Pittsburg, PA (April 1973).
13. S. Mukherjee and V. Kumar, Numerical analysis of time dependent inelastic deformation in metallic media using the boundary-integral equation method. ERDA Rep. No. COO-2733-14. Dept. Theoretical and Appl. Mech., Cornell Univ. Ithaca, New York (Dec. 1977).